9.6 Matching Pennies: A Game with No Pure Strategy Equilibrium

Example 9.2 (Matching Pennies). Two players simultaneously choose Heads or Tails.

		Player B	
		Heads	Tails
Player A	Heads	+1, −1	−1, +1
	Tails	−1, +1	+1, −1

Step 1: Why no pure strategy Nash equilibrium exists

Consider every candidate pure-strategy profile:
- **(H, H):** A gets +1, B gets −1. B deviates to T ⇒ B gets +1. *Not a NE.*
- **(H, T):** A gets −1, A deviates to T ⇒ A gets +1. *Not a NE.*
- **(T, H):** A gets −1, A deviates to H ⇒ A gets +1. *Not a NE.*
- **(T, T):** B gets −1, B deviates to H ⇒ B gets +1. *Not a NE.*

In every case, one player wants to deviate. **No pure strategy NE exists.**

Step 2: Finding the mixed strategy equilibrium via indifference

Let p = probability A plays Heads, and q = probability B plays Heads. **Making Player B indifferent.**

A chooses p so that B's expected payoff from Heads equals that from Tails:

$$\underbrace{p(-1)}_{\text{B plays H, A plays H}} + \underbrace{(1-p)(+1)}_{\text{B plays H, A plays T}} = \underbrace{p(+1)}_{\text{B plays T, A plays H}} + \underbrace{(1-p)(-1)}_{\text{B plays T, A plays T}}$$

$$-p + 1 - p = p - 1 + p \implies 1 - 2p = 2p - 1 \implies \boxed{p = \tfrac{1}{2}}$$

By symmetry, B makes A indifferent by setting $\boxed{q = \tfrac{1}{2}}$.

Step 3: Expected payoffs in equilibrium

Outcome	Probability	Payoff$_A$	Payoff$_B$
(H, H)	$\frac{1}{4}$	+1	−1
(H, T)	$\frac{1}{4}$	−1	+1
(T, H)	$\frac{1}{4}$	−1	+1
(T, T)	$\frac{1}{4}$	+1	−1
Expected		**0**	**0**

Neither player can improve by changing strategy unilaterally. Both earn **zero** in expectation—this is a *fair game.*

Key Point: Why randomize?

Randomization prevents your opponent from exploiting predictable behavior. Empirical evidence confirms this: studies of professional soccer penalty kicks (Chiappori, Levitt, & Groseclose, 2002) and tennis serves (Walker & Wooders, 2001) show that professionals mix at rates remarkably close to equilibrium predictions. *Discussion:* Both players earn 0 in expectation. Would you pay to enter this game?